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Dynamical symmetries in constrained systems: a Lagrangian analysis

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An intrinsic definition for dynamical symmetries on the velocity space TQ is proposed when the Lagrangian function is degenerate. A necessary and sufficient condition is provided so that vector fields of $\mathcal{X}(TQ)$, connected with constants of the motion, generate dynamical symmetries on the final constraint submanifold.

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1. Introduction

Recent papers on degenerate Lagrangian systems [1–5] have dealt with the question of the equivalence between the Lagrangian and the Hamiltonian approaches. In particular, in refs. [3,4] two results have been stressed: a definite relationship between Lagrangian and Hamiltonian constraints and a complete equivalence between the Lagrange and the Hamilton–Dirac equations. Nevertheless, a relevant disparity between the two formulations arises if one looks at the dynamical symmetry transformations (DST), at the link between constants of the motion and symmetries and at the introduction of gauge symmetries (for the canonical formulation see refs. [6–9]).

In the present work it is underlined how a suitable definition of dynamical symmetry in the phase space T^*Q must take into account or refuse the so-called "Dirac conjecture" (on this subject see refs. [10–13]) and we will comment on the consequences of both choices. Moreover, we will propose an intrinsic way of characterizing the DST within the Lagrangian formulation. To this purpose, as will be seen below, a suitable tool is a vector field satisfying the following properties: it must be tangent to the final constraint submanifold and it must map integral curves of the motion into equivalent integral curves.

This feature takes into account the fact that the motion takes place on the constraint submanifold and that, given the initial conditions, the equations of motion do not have a single solution in the general case.

The connection with the Hamiltonian analysis can be seen by defining on TQ (the tangent space of the configuration space Q) a particular class of vector fields which exhibit prominent analogies with the Hamiltonian vector fields of $\mathcal{X}(T^*Q)$. We will show that the acceptance of the Dirac conjecture guarantees a one-to-one correspondence between the DST of TQ and the ones of the phase space T^*Q; viceversa, by refusing it, an entire class of DST of TQ has no correspondence in T^*Q.

In the present work emphasis will be placed on the DST belonging to the orthogonal complement of $\omega_{\mathcal{L}}$ (the presymplectic 2-form defined on TQ through the Lagrangian). The reason for this choice lies in the close link between such vector fields and the first class constraints of T^*Q, which play an essential role in building gauge transformations [10].

To avoid any pathologies and to get results which hold as global statements, we assume the standard hypotheses in the constraint theory (for instance, that the Legendre mapping is a submersion onto its image with connected fibers; that constraint submanifolds are closed and embedded in T^*Q; that the rank of the Poisson brackets of constraints is constant). More specific hypotheses may occur and are specified in relation with particular results.

2. Dynamical symmetries in phase space

Let $\mathcal{L}$ be the Lagrangian for a mechanical system with n degrees of freedom. As is well known [10], if the Lagrangian is degenerate and

$$\operatorname{rank}\left|\partial^2\mathcal{L}/\partial\dot{q}\,\partial\dot{q}\right| = n - m, \tag{2.1}$$

m primary Hamiltonian constraints exist in T^*Q, and at first m undetermined multipliers appear in the Hamilton–Dirac equations. One has a submanifold $M_0 \subset T^*Q$ defined by the primary constraints

$$\overset{(0)}{\phi}_\mu = 0, \quad \mu = 1, m, \tag{2.2}$$

for which compatibility conditions with the equations of motion arise. Therefore, the conservation of the constraints must be imposed by requiring:

$$j^*_{M_0}(\{\overset{(0)}{\phi}_\mu, H\} + \lambda^\nu\{\overset{(0)}{\phi}_\mu, \overset{(0)}{\phi}_\nu\}) = 0, \quad \mu = 1, m, \tag{2.3}$$

where $j_{M_0} : M_0 \hookrightarrow T^*Q$ is the identification mapping, H is such that $F\mathcal{L}^* H = E_\mathcal{L}$ ($E_\mathcal{L}$ is the Lagrangian energy, $F\mathcal{L}$ the fibre derivative of the Lagrangian function) and finally, $\lambda^\mu \in \mathcal{F}(T^*Q)$ are the Lagrange multipliers.

Conditions (2.3) can give rise to: (i) the determination of $m-m_1$ multipliers, if $m-m_1$ is the rank of the matrix $|\{\phi_\mu^{(0)},\phi_\nu^{(0)}\}|$; (ii) secondary constraints; (iii) identities on M_0.

Now, we will summarize the analysis of the constraints following the notations used in ref. [3].

It is known that one can construct a maximal set of m_1 $(m_1 \le m)$ constraints $\phi_{\mu_0}^{(0)}$ which are first class on M_0, i.e,

$$j^*_{M_0}\{\phi^{(0)}_{\mu_0},\phi^{(0)}_{\mu}\} = 0, \quad \mu_0 = 1, m_1; \quad \mu = 1, m. \tag{2.4}$$

The remaining independent $m-m_1$ constraints satisfy

$$\det|\{\phi^{(0)}_{\mu'_0},\phi^{(0)}_{\nu'_0}\}| \neq 0, \quad \mu'_0, \nu'_0 = 1, m-m_1, \tag{2.5}$$

and are associated with determined multipliers so that, defining

$$H^{(1)} = H + \lambda^{\mu'_0}\phi^{(0)}_{\nu'_0}, \tag{2.6}$$

the following property holds true:

$$j^*_{M_0}\{\phi^{(0)}_{\mu'_0},H^{(1)}\} = 0, \quad \mu'_0 = 1, m-m_1. \tag{2.7}$$

In this way we see that secondary constraints are needed only if we apply conditions (2.3) to a function $\phi_{\mu_0}^{(0)}$ and find that it is not satisfied. Otherwise, if the condition becomes an identity, we are in case (iii). However, for the sake of simplicity and since this does not affect calculations, we will call secondary constraints all the functions

$$\phi^{(1)}_{\mu_0} = \{\phi^{(0)}_{\mu_0},H^{(1)}\}, \quad \mu_0 = 1, m_1. \tag{2.8}$$

One must only keep in mind that the submanifold $M_1 \subset M_0$, which is defined by (2.2) together with

$$\phi^{(1)}_{\mu_0} = 0, \quad \mu_0 = 1, m_1, \tag{2.9}$$

can possibly have dimensions greater than $2n-m-m_1$. We use a similar convention for higher order constraints as well.

Then, having linearized possible quadratic constraints, the second step of the consistency analysis can start: one requires the conservation of the secondary constraints

$$j^*_{M_1}(\{\phi^{(1)}_{\mu_0},H^{(1)}\} + \lambda^{\mu_0}\{\phi^{(1)}_{\mu_0},\phi^{(0)}_{\nu_0}\}) = 0, \quad \mu_0 = 1, m_1. \tag{2.10}$$

At this point, it is possible to choose m_2 ($m_2 \leq m_1$) constraints $\phi^{(0)}_{\mu_1}$ from among the functions $\phi^{(0)}_{\mu_0}$ in such a way that they are first class also on M_1:

$$j^*_{M_1}\{\phi^{(0)}_{\mu_1}, \phi^{(1)}_{\mu_0}\} = 0, \quad \mu_1 = 1, m_2; \quad \mu_0 = 1, m_1. \tag{2.11}$$

Again, the remaining $m_1 - m_2$ constraints are such that

$$\det|\{\phi^{(0)}_{\mu'_1}, \phi^{(1)}_{\nu'_1}\}| \neq 0, \quad \mu'_1, \nu'_1 = 1, m_1 - m_2, \tag{2.12}$$

where $\phi^{(1)}_{\nu'_1}$ are secondary second class constraints. The remaining m_2 secondary constraints can be written in such a way that their PBs with the functions $\phi^{(0)}_{\mu_0}$ vanish on M_1.

From (2.12) we see that (2.10) leads to a determination of $m_1 - m_2$ multipliers so that the Hamiltonian

$$H^{(2)} = H^{(1)} + \lambda^{\mu'_1}\phi^{(0)}_{\mu'_1} \tag{2.13}$$

satisfies

$$j^*_{M_1}\{\phi^{(1)}_{\mu'_1}, H^{(2)}\} = 0, \quad \mu'_1 = 1, m_1 - m_2. \tag{2.14}$$

On the contrary, the functions

$$\phi^{(2)}_{\mu_1} = \{\phi^{(1)}_{\mu_1}, H^{(2)}\}, \quad \mu_1 = 1, m_2, \tag{2.15}$$

may be tertiary constraints. The procedure is clearly iterative from this point on: one defines M_2 and studies on it the PBs between primary and tertiary constraints which form the matrix $|\{\phi^{(2)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}|$, then one finds primary first class constraints $\phi^{(0)}_{\mu_2}$, with $\mu_2 = 1, m_3$ ($m_3 \leq m_2$), such that

$$j^*_{M_2}\{\phi^{(0)}_{\mu_2}, \phi^{(2)}_{\mu_1}\} = 0, \quad \mu_1 = 1, m_2, \tag{2.16}$$

and new Lagrange multipliers, whose number is $m_2 - m_3$.

The analysis is finished at this step only if the conservation of the constraints $\phi^{(2)}_{\mu_2}$ (linearized if needed) is automatically satisfied on M_2:

$$j^*_{M_2}\{\phi^{(2)}_{\mu_2}, H^{(3)}\} = 0, \quad \mu_2 = 1, m_3, \tag{2.17}$$

where the final Hamiltonian $H^{(3)} = H^{(2)} + \lambda^{\mu'_2}\phi^{(0)}_{\mu'_2}$ is first class on M_2.

If one writes the equations of motion by means of $H^{(3)}$, once the initial conditions on M_2 are chosen the solutions lie on M_2. We note that in this algorithm the constraints which give rise to the following generation are not necessarily first class except for the $\phi^{(0)}_{\mu_0}$ at the first step; however, it is possible to define secondary and tertiary constraints

$$\overline{\phi}^{(1)}_{\mu_2} = \{\phi^{(0)}_{\mu_2}, H^{(3)}\}, \quad \mu_2 = 1, m_3, \tag{2.18}$$

$$\overline{\phi}^{(2)}_{\mu_2} = \{\overline{\phi}^{(1)}_{\mu_2}, H^{(3)}\}, \quad \mu_2 = 1, m_3, \tag{2.19}$$

which are first class on M_2 for known involutive properties of the first class functions. However, such constraints may be quadratic. Indeed we want to underline that a subdivision in first and second class is possible only *a posteriori*, that is, on the final submanifold and once the first class Hamiltonian has been found.

In this work we limit the study to the case in which no tertiary constraints appear. Therefore M_1 is the final submanifold, i.e.,

$$j^*_{M_1}\{\phi^{(1)}_{\mu_0}, H^{(2)}\} = 0, \quad \mu_0 = 1, m_1, \tag{2.20}$$

where $H^{(2)}$ is first class on M_1; the secondary first class constraints are

$$\overline{\phi}^{(1)}_{\mu_1} = \{\phi^{(0)}_{\mu_1}, H^{(2)}\} = 0, \quad \mu_1 = 1, m_2. \tag{2.21}$$

In order to write the equations of motion on the final submanifold, the equation

$$i_{\Gamma'} j^*_{M_1} \Omega = j^*_{M_1} dH \tag{2.22}$$

must be solved for $\Gamma' \in \mathcal{X}(M_1)$; here Ω is the canonical symplectic two-form and the Hamiltonian H is defined except for primary constraints. Using X_f to indicate the Hamiltonian vector field belonging to $\mathcal{X}(T^*Q)$ which is the solution for

$$i_{X_f}\Omega = df, \quad f \in \mathcal{F}(T^*Q), \tag{2.23}$$

we can take (see ref. [2]) the most general solution for (2.22) and write a dynamics $\Gamma \in \mathcal{X}(T^*Q)$ such that $\Gamma|_{M_1} = j_{M_1 *}\Gamma'$,

$$\Gamma = X_{H^{(2)}} + \lambda^{\mu_1} X_{\phi^{(0)}_{\mu_1}} + \xi^{\mu_1} X_{\overline{\phi}^{(1)}_{\mu_1}}, \tag{2.24}$$

with λ^{μ_1} and ξ^{μ_1} arbitrary functions of $\mathcal{F}(T^*Q)$.

It has been observed [2,7] that, when complete equivalence between Hamiltonian and Lagrangian descriptions of motion is required, solution (2.24) cannot be accepted: thus the multipliers ξ^{μ_1} should be equal to zero in (2.24) whenever a vector field is sought which is also solution on $M_0 = F\mathcal{L}(TQ)$. In fact, it can easily be demonstrated that only the field

$$\Gamma = X_{H^{(2)}} + \lambda^{\mu_1} X_{\phi^{(0)}_{\mu_1}} \tag{2.25}$$

is also a solution for the equation

$$j^*_{M_0} i_\Gamma \Omega = j^*_{M_0} dH. \tag{2.26}$$

On the contrary, retaining (2.24) as the solution is the equivalent of extending the total Hamiltonian, including in it secondary first class constraints; this is the content of the Dirac conjecture. We are going to examine how the choice of the dynamics affects the definition of dynamical symmetry.

Let us define the sets $\mathcal{X}(M_1)^\perp$ and $\underline{\mathcal{X}(M_1)}$:

$$\mathcal{X}(M_1)^\perp \equiv \{X \in \mathcal{X}(T^*Q) \setminus j^*_{M_1} i_X \Omega = 0\}, \tag{2.27}$$

$$\underline{\mathcal{X}(M_1)} \equiv \{X \in \mathcal{X}(T^*Q) \setminus X|_{M_1} = j_{M_1 *} X', X' \in \mathcal{X}(M_1)\}. \tag{2.28}$$

The choice of dynamics (2.25) rather than (2.24) leads to consequences which are relevant from the dynamical symmetry point of view. To see this we can study the algebra of the set $\underline{\mathcal{X}(M_1)}^\perp = \underline{\mathcal{X}(M_1)} \cap \mathcal{X}(M_1)^\perp$. It is evident that, given $X \in \underline{\mathcal{X}(M_1)}^\perp$ and a Hamiltonian vector field $Y \in \underline{\mathcal{X}(M_1)}$, the vector field $[X, Y]$ belongs to $\underline{\mathcal{X}(M_1)}^\perp$. The vector fields $X_{\phi^{(0)}_{\mu_1}}$ and $X_{\bar{\phi}^{(1)}_{\mu_1}}$ belong to $\underline{\mathcal{X}(M_1)}^\perp$ because $\phi^{(0)}_{\mu_1}$ and $\overline{\phi}^{(1)}_{\mu_1}$ are first class functions defining M_1.

Let us take a function G, which is (on M_1) a constant of the motion of first class. We get $X_G \in \underline{\mathcal{X}(M_1)}$ and

$$j^*_{M_1} \{G, H^{(2)}\} = 0, \tag{2.29}$$

$$j^*_{M_1} \{G, \phi^{(0)}_{\mu_1}\} = 0, \quad \mu_1 = 1, m_2, \tag{2.30}$$

$$j^*_{M_1} \{G, \overline{\phi}^{(1)}_{\mu_1}\} = 0, \quad \mu_1 = 1, m_2. \tag{2.31}$$

All the Poisson brackets in (2.29), (2.30), (2.31) are combinations of first class constraints (primary and secondary) because of the involutive properties. Then, taking $\Gamma \in \underline{\mathcal{X}(M_1)}$ as in (2.25), we have

$$[\Gamma, X_G] = X_{\{G,H^{(2)}\}} + \lambda^{\mu_1} X_{\{G,\phi^{(0)}_{\mu_1}\}} + \{G, \lambda^{\mu_1}\} X_{\phi^{(0)}_{\mu_1}}. \tag{2.32}$$

Using (2.24) we obtain an analogous result:

$$[\Gamma, X_G] = X_{\{G,H^{(2)}\}} + \lambda^{\mu_1} X_{\{G,\phi^{(0)}_{\mu_1}\}} + \{G,\lambda^{\mu_1}\} X_{\phi^{(0)}_{\mu_1}} + \xi^{\mu_1} X_{\{G,\bar{\phi}^{(1)}_{\mu_1}\}} + \{G,\xi^{\mu_1}\} X_{\bar{\phi}^{(1)}_{\mu_1}}. \tag{2.33}$$

Using what has been mentioned above one sees immediately that in both cases $[\Gamma, X_G]$ belongs to $\underline{\mathcal{X}(M_1)^{\perp}}$. Indeed, these commutators yield the variation in dynamics Γ under infinitesimal canonical transformations generated by G. On the other hand, (2.24) and (2.25) do not represent a unique solution but rather an entire equivalence class of vector fields with two different equivalence relationships: in (2.24)

$$\Gamma_2 \sim \Gamma_1 \Longleftrightarrow \Gamma_2 = \Gamma_1 + \lambda^{\mu_1} X_{\phi^{(0)}_{\mu_1}} + \xi^{\mu_1} X_{\bar{\phi}^{(1)}_{\mu_1}}, \tag{2.34}$$

whereas in (2.25)

$$\Gamma_2 \sim \Gamma_1 \Longleftrightarrow \Gamma_2 = \Gamma_1 + \lambda^{\mu_1} X_{\phi^{(0)}_{\mu_1}}. \tag{2.35}$$

Then, a natural way of characterizing a dynamical symmetry is to require

$$[\Gamma, X_G]|_{M_1} \sim 0. \tag{2.36}$$

If one accepts Dirac's conjecture, using the equivalence relationship (2.34) together with (2.33) it can immediately be inferred that every first class constant of the motion generates a DST.

On the other hand, the class of solutions (2.25) is not always invariant under transformations generated by first class constants of the motion because of (2.32) and (2.35). Such an invariance is satisfied using (2.25) as the dynamics if, and only if, a set of functions $a^{\mu_1}, b^{\mu_1}_{\nu_1} \in \mathcal{F}(T^*Q)$ exists such that

$$X_{\{G,H^{(2)}\}}|_{M_1} = (j^*_{M_1} a^{\mu_1}) X_{\phi^{(0)}_{\mu_1}}|_{M_1}, \tag{2.37}$$

$$X_{\{G,\phi^{(0)}_{\nu_1}\}}|_{M_1} = (j^*_{M_1} b^{\mu_1}_{\nu_1}) X_{\phi^{(0)}_{\mu_1}}|_{M_1}, \quad \nu_1 = 1, m_2. \tag{2.38}$$

In this case X_G is a DST. Conditions (2.37) and (2.38), thought to be conditions for G, are equivalent to the one given in ref. [7].

Since it is known that gauge generators in T^*Q must be constraints, the case in which $G = \phi^{(0)}_{\mu_1}$, so that (2.38) is automatically satisfied, is particularly interesting.

One can note that a sufficient condition for (2.37) is in this case

$$j^*_{M_0} \bar{\phi}^{(1)}_{\mu_1} = 0 \tag{2.39}$$

(that is the situation in which dim $M_1 > 2n - m - m_1$). Then we have

$$F\mathcal{L}^* \overline{\phi}^{(1)}_{\mu_1} = 0, \tag{2.40}$$

which is an identity in TQ. Therefore, as will be demonstrated in section 4, not all primary Lagrangian constraints are independent and a symmetry for the Lagrangian function exists in TQ.

3. Degeneracy of $\omega_{\mathcal{L}}$ on the final constraint submanifold

Analysis of gauge symmetries in TQ shows relevant differences from what was mentioned in the previous section. This fact is related to the particular geometrical properties of the fiber bundle TQ. It is known [14] that the vertical endomorphism S plays a central role in building the intrinsic structure of TQ. S is a (1–1) tensor field whose Nijenhuis tensor is equal to zero. Seen as a mapping for vector fields, S is such that its kernel coincides with its image. Use will be made herein of the well-known Liouville vector field $\Delta \in \mathcal{X}(TQ)$ and of second order vector fields $\Gamma_0 \in \mathcal{X}(TQ)$, with the property $S(\Gamma_0) = \Delta$. Given a Lagrangian function $\mathcal{L} \in \mathcal{F}(TQ)$, the Lagrangian two-form is

$$\omega_{\mathcal{L}} = -d(d\mathcal{L} \circ S) \tag{3.1}$$

and the energy function is

$$E_{\mathcal{L}} = L_{\Delta}\mathcal{L} - L. \tag{3.2}$$

The Lagrange equations of motion take the form

$$L_{\Gamma}(d\mathcal{L} \circ S) = d\mathcal{L}. \tag{3.3}$$

A comparison between the formulations in TQ and in T^*Q would be very useful and is implemented by the first order differential operator K [3] and by the mapping $R(L)_w : T_{F\mathcal{L}(w)}T^*Q \longrightarrow T_wTQ$ (w being a vector of TQ), defined in ref. [15]. The action of the operator K is given by

$$K(f) = F\mathcal{L}^*\{f, H\} + v^\mu F\mathcal{L}^*\{f, \overset{(0)}{\phi}_\mu\}, \quad f \in \mathcal{F}(T^*Q), \tag{3.4}$$

with v^μ known functions of $\mathcal{F}(TQ)$. The operator takes a function in T^*Q and gives its time derivate as a function in TQ.

With regard to the mapping $R(L)_w$, it must be pointed out that no corresponding global mapping exists from $\mathcal{X}(T^*Q)$ to $\mathcal{X}(TQ)$. Using $R(L)_w$, the most we can do is take a field of $\mathcal{X}(T^*Q)$ restricted to M_0 in order to construct a field of $\mathcal{X}(TQ)$. Nevertheless, this being implied, we will write

$R(L)$ instead of $R(L)_w$ following the notation of ref. [15]. Use will be made of the property

$$R(L)X_{\phi_\mu^{(0)}} = S(Z_\mu) = K_\mu^v \in V(\ker\omega_{\mathcal{L}}), \quad \mu = 1, m, \tag{3.5}$$

where $V(\ker\omega_{\mathcal{L}}) = \ker\omega_{\mathcal{L}} \cap V(TQ)$ with

$$\ker\omega_{\mathcal{L}} \equiv \{X \in \mathcal{X}(TQ) \setminus i_X\omega = 0\}, \tag{3.6}$$

$$V(TQ) \equiv \{X \in \mathcal{X}(TQ) \setminus S(X) = 0\}. \tag{3.7}$$

The local expression of the vector fields K_μ^v is given in appendix A. The vector fields Z_μ are defined modulo arbitrary vertical fields. It has been demonstrated [4] that, if ϕ is any Hamiltonian constraint, all Lagrangian constraints can be obtained by posing

$$K(\phi) = 0 \tag{3.8}$$

In the present case the following relations hold:

$$\chi_{\mu_0}^{(1)} = K(\phi_{\mu_0}^{(0)}), \quad \mu_0 = 1, m_1, \tag{3.9}$$

$$\chi_{\mu'_0}^{(1)} = K(\phi_{\mu'_0}^{(0)}), \quad \mu'_0 = 1, m - m_1. \tag{3.10}$$

The functions $\chi_{\mu_0}^{(1)}$ give the so-called dynamical constraints. These define the submanifold S'_1, which is minimal in order to write the equations of motion. On S'_1 at least one solution $\Gamma \in \mathcal{X}(TQ)$ exists for the equation

$$j_{S'_1}^* i_\Gamma \omega_{\mathcal{L}} = j_{S'_1}^* dE. \tag{3.11}$$

By requiring the second order character for Γ, the so-called SODE constraints arise, corresponding to the functions $\chi_{\mu'_0}^{(1)}$. In this manner the primary constraint submanifold S_1 is obtained.

By requiring the tangency of Γ to S_1, the secondary constraints $\chi_{\mu'_1}^{(2)}$ arise. The following relation holds:

$$j_{S_1}^* \chi_{\mu'_1}^{(2)} = j_{S_1}^* K(\phi_{\mu'_1}^{(1)}), \quad \mu'_1 = 1, m_1 - m_2. \tag{3.12}$$

Since there are no tertiary Hamiltonian constraints, we have $j_{S_1}^* K(\phi_{\mu_1}^{(1)}) = 0$ for $\mu_1 = 1, m_2$. The constraints $\chi_{\mu'_1}^{(2)}$, together with all the others, define the final submanifold S_2 . Only the dynamical constraints are $F\mathcal{L}$-projectable [3].

In order to write dynamical symmetries in TQ the set $\underline{\mathcal{X}(S_2)}^\perp = \mathcal{X}(S_2)^\perp \cap \underline{\mathcal{X}(S_2)}$ must be studied. This set is defined similarly to $\underline{\mathcal{X}(M_1)}^\perp$ in the previous section. The elements of $\underline{\mathcal{X}(S_2)}^\perp$ can be correlated to those of $\underline{\mathcal{X}(M_1)}^\perp$ although not one to one. To do so we begin by looking for an $F\mathcal{L}$-projectable basis for the set $\ker\omega_\mathcal{L} \cap \underline{\mathcal{X}(S_2)}$. In fact, this set is contained in $\underline{\mathcal{X}(S_2)}^\perp$ because of the relation

$$\ker\omega_\mathcal{L} = \mathcal{X}(TQ)^\perp \subseteq \mathcal{X}(S_2)^\perp. \tag{3.13}$$

An $F\mathcal{L}$-projectable basis for $\ker\omega_\mathcal{L}$ exists and may be made of the m vector fields K_μ^v, defined in (3.5), together whith the m_1 vector fields $K_{\mu_0} \in \mathcal{X}(TQ)$ such that

$$F\mathcal{L}_* K_{\mu_0} = X_{\phi^{(0)}_{\mu_0}}, \quad \mu_0 = 1, m_1. \tag{3.14}$$

Condition (3.14) does not characterize the fields K_{μ_0} in an unambiguous manner. In fact, since $\ker F\mathcal{L}_* = V(\ker\omega_\mathcal{L})$, we can add any element of $V(\ker\omega_\mathcal{L})$ to K_{μ_0}. A particular choice of K_{μ_0} can be made so that

$$L_{K_{\mu_0}} v^\mu = 0, \quad \mu = 1, m; \quad \mu_0 = 1, m_1. \tag{3.15}$$

The local expression of this basis is given in appendix A.

The tangency of $V(\ker\omega_\mathcal{L})$ to the final constraint submanifold was checked in ref. [3]. The result is that all the vector fields of $V(\ker\omega_\mathcal{L})$ are tangent to S_1', defined by

$$L_{K_{\mu_0}} E_\mathcal{L} = 0, \quad \mu_0 = 1, m_1. \tag{3.16}$$

Furthermore, only the fields $K^v_{\mu_0} = S(K_{\mu_0})$ are tangent to S_1 and only the fields $K^v_{\mu_1}$ with $\mu_1 = 1, m_2$ are tangent to S_2. With regard to the other fields in our basis, the condition of tangency to S_1' is

$$\begin{aligned} 0 &= j^*_{S_1'} L_{K_{\mu_0}} (L_{K_{\nu_0}} E_\mathcal{L}) = j^*_{S_1'} F\mathcal{L}^* \{\phi^{(0)}_{\mu_0}, \{\phi^{(0)}_{\nu_0}, H^{(1)}\}\} \\ &= j^*_{S_1'} F\mathcal{L}^* \{\phi^{(0)}_{\mu_0}, \phi^{(1)}_{\nu_0}\}, \quad \mu_0, \nu_0 = 1, m_1. \end{aligned} \tag{3.17}$$

Thus, in a manner similar to what happens in T^*Q, the vector fields K_{μ_0} are split into $K_{\mu_1'}$ ($\mu_1' = 1, m_1 - m_2$) and K_{μ_1} ($\mu_1 = 1, m_2$), the latter being tangent to S_1'. In fact,

$$F\mathcal{L}_* K_{\mu_1} = X_{\phi^{(0)}_{\mu_1}}, \quad \mu_1 = 1, m_2, \tag{3.18}$$

and $X_{\phi^{(0)}_{\mu_1}}$ is tangent to $M_1 = F\mathcal{L}(S_1')$.

It is always possible to construct m_2 vector fields $\overline{K}_{\mu_1}$ which are tangent to S_2. The conditions are

$$j^*_{S_2} L_{\overline{K}_{\mu_1}} (v^{\mu'_0} - F\mathcal{L}^* \lambda^{\mu'_0}) = 0, \quad \mu'_0 = 1, m - m_1, \tag{3.19}$$

$$j^*_{S_2} L_{\overline{K}_{\mu_1}} (v^{\mu'_1} - F\mathcal{L}^* \lambda^{\mu'_1}) = 0, \quad \mu'_1 = 1, m_1 - m_2. \tag{3.20}$$

Here the expression for non-$F\mathcal{L}$-projectable constraints given in ref. [4] has been used. The geometrical idea is to add a linear combination of non-tangent vector fields belonging to $V(\ker \omega_{\mathcal{L}})$ to every K_{μ_1} which is not tangent to S_2, thus obtaining a tangent vector field still in $\ker \omega_{\mathcal{L}}$ while being $F\mathcal{L}$-projectable. Let

$$C^{\nu'_0}_{\mu_1} K^v_{\nu'_0} + C^{\nu'_1}_{\mu_1} K^v_{\nu'_1}, \quad C^{\nu'_0}_{\mu_1}, C^{\nu'_1}_{\mu_1} \in \mathcal{F}(TQ), \tag{3.21}$$

be the linear combination which must be added to K_{μ_1}. Using (3.15) and the property

$$L_{K^v_\mu} v^\nu = \delta^\nu_\mu, \quad \mu, \nu = 1, m, \tag{3.22}$$

from (3.19) and (3.20) we get

$$j^*_{S_2} C^{\nu'_0}_{\mu_1} = j^*_{S_2} F\mathcal{L}^* \{\lambda^{\nu'_0}, \phi^{(0)}_{\mu_1}\}, \quad \nu'_0 = 1, m - m_1, \tag{3.23}$$

$$j^*_{S_2} C^{\nu'_1}_{\mu_1} = j^*_{S_2} F\mathcal{L}^* \{\lambda^{\nu'_1}, \phi^{(0)}_{\mu_1}\}, \quad \nu'_1 = 1, m_1 - m_2. \tag{3.24}$$

(We recall that $\lambda^{\nu'_0}$ and $\lambda^{\nu'_1}$ are known functions of $\mathcal{F}(T^*Q)$, determined on M_0 and on M_1, respectively, so the PB can be unambiguously calculated.) In this way, the following have been proved:

Proposition 3.1. *An $F\mathcal{L}$-projectable basis for* $\ker \omega_{\mathcal{L}} \cap \underline{\mathcal{X}(S_2)}$ *is made of the vector fields*

$$K^v_{\mu_1} = R(L) X_{\phi^{(0)}_{\mu_1}}, \quad \mu_1 = 1, m_2,$$

and of the vector fields $\overline{K}_{\mu_1}$ with $\mu_1 = 1, m_2$, determined by the condition

$$F\mathcal{L}_* \overline{K}_{\mu_1} = X_{\phi^{(0)}_{\mu_1}}, \quad \mu_1 = 1, m_2,$$

together with conditions

$$L_{\overline{K}_{\mu_1}} v^{\nu_1} = 0,$$

$$L_{\overline{K}_{\mu_1}} v^{\nu'_0} = F\mathcal{L}^* \{\lambda^{\nu'_0}, \phi^{(0)}_{\mu_1}\}, \qquad L_{\overline{K}_{\mu_1}} v^{\nu'_1} = F\mathcal{L}^* \{\lambda^{\nu'_1}, \phi^{(0)}_{\mu_1}\},$$

where $\nu_1 = 1, m_2$; $\nu'_0 = 1, m - m_1$; $\nu'_1 = 1, m_1 - m_2$.

The relation between $\overline{K}_{\mu_1}$ and K_{μ_1} is:

$$\overline{K}_{\mu_1} = K_{\mu_1} + F\mathcal{L}^*\{\lambda^{\nu'_0}, \overset{(0)}{\phi}_{\mu_1}\}R(L)X_{\overset{(0)}{\phi}_{\nu'_0}} + F\mathcal{L}^*\{\lambda^{\nu'_1}, \overset{(0)}{\phi}_{\mu_1}\}R(L)X_{\overset{(0)}{\phi}_{\nu'_1}}. \quad (3.25)$$

However, there are other independent vector fields in $\underline{\mathcal{X}(S_2)}^{\perp}$ apart from those defined in proposition 3.1. If $\Gamma \in \underline{\mathcal{X}(S_2)}$ is the second order dynamics tangent to S_2 we have

$$j^*_{S_2} i_{[\Gamma,\overline{K}_{\mu_1}]}\omega_{\mathcal{L}} = j^*_{S_2}(i_\Gamma L_{\overline{K}_{\mu_1}}\omega_{\mathcal{L}} - L_{\overline{K}_{\mu_1}} i_\Gamma\omega_{\mathcal{L}}) = -dj^*_{S_2} L_{\overline{K}_{\mu_1}} E_{\mathcal{L}} = 0. \quad (3.26)$$

Therefore the vector field $[\Gamma, \overline{K}_{\mu_1}]$ also belongs to $\underline{\mathcal{X}(S_2)}^{\perp}$. Using the property [4]

$$j^*_{S_1} K(f) = j^*_{S_1} L_\Gamma(F\mathcal{L}^* f), \quad f \in \mathcal{F}(T^*Q), \quad (3.27)$$

we get

$$[\Gamma, \overline{K}_{\mu_1}]|_{S_2} = \overline{Y}_{\mu_1}|_{S_2} + (j^*_{S_2} v^{\nu_1})[\overline{K}_{\nu_1}, \overline{K}_{\mu_1}]|_{S_2} - (j^*_{S_2} L_{\overline{K}_{\mu_1}} L_\Gamma v^{\nu_1})K^v_{\nu_1}|_{S_2}, \quad (3.28)$$

where the vector field $\overline{Y}_{\mu_1}$, obviously belonging to $\underline{\mathcal{X}(S_2)}^{\perp}$, is associated with the field Y_{μ_1} such that

$$F\mathcal{L}_* Y_{\mu_1} = X_{\overline{\phi}^{(0)}_{\mu_1}}, \quad \mu_1 = 1, m_2, \quad (3.29)$$

as $\overline{K}_{\mu_1}$ was associated with K_{μ_1}. In general, $\overline{Y}_{\mu_1}$ does not belong to $\ker\omega_{\mathcal{L}}$, likewise $X_{\overline{\phi}^{(0)}_{\mu_1}}$ does not belong to $\underline{\mathcal{X}(M_0)}^{\perp}$. Calculation of the Lie bracket (3.28) is given in appendix B as is the local expression for $\overline{Y}_{\mu_1}$.

All the results concerning K_{μ_1} and Y_{μ_1} can be generalized. To do so one can associate a particular vector field $X_{(G)} \in \mathcal{X}(TQ)$ with any function $G \in \mathcal{F}(T^*Q)$ for which

$$j^*_{M_1}\{G, \overset{(0)}{\phi}_\mu\} = 0, \quad \mu = 1, m, \quad (3.30)$$

holds true. The field $X_{(G)}$ must satisfy the following properties:

$$R(L)X_G = S(X_{(G)}), \quad (3.31)$$

$$[X_{(G)}, K^v_\mu] = 0, \quad \mu = 1, m, \quad (3.32)$$

$$F\mathcal{L}_* X_{(G)}|_{M_1} = X_G|_{M_1}, \quad (3.33)$$

$$L_{X_{(G)}} v^\mu = 0, \quad \mu = 1, m. \quad (3.34)$$

(3.32) is the condition for $X_{(G)}$ to be $F\mathcal{L}$-projectable; from (3.33) it follows that

$$j^*_{S'_1} i_{X_{(G)}} \omega = j^*_{S'_1} dF\mathcal{L}^* G, \tag{3.35}$$

and (3.34) is useful in the construction of the vector field

$$\overline{X}_{(G)} = X_{(G)} + F\mathcal{L}^*\{\lambda^{\mu'_0}, G\}K^v_{\mu'_0} + F\mathcal{L}^*\{\lambda^{\mu'_1}, G\}K^v_{\mu'_1}, \tag{3.36}$$

which is tangent to S_2. A local expression of $X_{(G)}$ can be found in appendix A.

If G is a first class constant of the motion on M_1, using (3.4) and (3.27) one obtains

$$j^*_{S'_1} L_\Gamma (F\mathcal{L}^* G) = 0, \tag{3.37}$$

whereas, taking into account that $j^*_{S'_1} L_\Gamma \omega = 0$, using (3.35) and (3.37) one gets

$$j^*_{S_2} i_{[\Gamma, \overline{X}_{(G)}]} \omega_{\mathcal{L}} = 0. \tag{3.38}$$

Finally, the generalization of expression (3.28) can be obtained

$$\begin{aligned}[\Gamma, \overline{X}_{(G)}]\big|_{S_2} &= \overline{X}_{(\{G, H^{(2)}\})}\big|_{S_2} \\ &\quad + (j^*_{S_2} v^{\nu_1})[\overline{K}_{\nu_1}, \overline{X}_{(G)}]\big|_{S_2} - (j^*_{S_2} L_{\overline{X}_{(G)}} L_\Gamma v^{\nu_1}) K^v_{\nu_1}\big|_{S_2}. \end{aligned} \tag{3.39}$$

The results for K_{μ_1} and Y_{μ_1} can be derived taking G equal to $\phi^{(0)}_{\mu_1}$ and $\overline{\phi}^{(1)}_{\mu_1}$, respectively.

4. Dynamical symmetries in TQ

If the Lagrangian function is degenerate, when we write the equation for $\Gamma \in \mathcal{X}(TQ)$,

$$i_\Gamma \omega_{\mathcal{L}} = dE_{\mathcal{L}}, \tag{4.1}$$

we must add to it the so-called SODE condition, $S(\Gamma) = \Delta$, in order to obtain Lagrange equations. Therefore, any infinitesimal transformation preserving the structure of the tangent bundle must at least be Newtonoid with respect to Γ. A vector field $X(\Gamma) \in \mathcal{X}(TQ)$ is said to be Newtonoid for Γ if

$$X(\Gamma) = X + S[\Gamma, X], \quad X \in \mathcal{X}(TQ). \tag{4.2}$$

As the property

$$S[X(\Gamma), \Gamma] = 0 \tag{4.3}$$

holds, the new dynamics will also be a second order vector field. Therefore a DST must have the form (4.2).

Now, let us return to the equation for $\Gamma \in \underline{X(S_2)}$ (only the final submanifold is relevant from the physical point of view):

$$j^*_{S_2} i_\Gamma \omega_L = j^*_{S_2} dE_L. \tag{4.4}$$

In TQ, as in T^*Q, the solution of (4.4) is not unique but rather is an equivalence class of vector fields, whose quotient is made by imposing that the elements of the set $\underline{\mathcal{X}(S_2)}^\perp \cap V(TQ)$ are equivalent to zero. As a matter of fact, since Γ must be a second order vector field, we can add to it only vertical fields belonging to $\underline{\mathcal{X}(S_2)}^\perp$. If, for instance, ω_L is of constant rank, that is

$$\underline{\mathcal{X}(S_2)}^\perp \cap V(TQ) = V(\ker \omega_L), \tag{4.5}$$

the equivalent solutions of (4.4) are characterized in the following way:

$$\Gamma_1 \sim \Gamma_2 \Longleftrightarrow \Gamma_2 = \Gamma_1 + \eta^{\mu_1} K^v_{\mu_1}, \tag{4.6}$$

with arbitrary functions η^{μ_1}.

Thus, a Lagrangian DST must satisfy

$$j^*_{S_2} i_{[X(\Gamma),\Gamma]} \omega_L = 0. \tag{4.7}$$

To this purpose we give the following

Definition 4.1. A Newtonoid vector field $X(\Gamma)$ is a *Lagrangian DST* if and only if

$$X(\Gamma) \in \underline{\mathcal{X}(S_2)}, \tag{4.8}$$

$$[X(\Gamma),\Gamma] \in \underline{\mathcal{X}(S_2)}^\perp. \tag{4.9}$$

A recurring problem in Lagrangian analysis is how to check the tangency of vector fields. Addressing this point we give the following results (see appendix C for proof).

Lemma 4.2. *Let f be a first class function on M_1. Then $R(L)X_f$ belongs to $\underline{\mathcal{X}(S_2)}$ if and only if*

$$j^*_{S_2} L_{Z_\mu(\Gamma)} F\mathcal{L}^* f = 0, \quad \mu = 1, m, \tag{4.10}$$

$$j^*_{S_2} L_{Y_{\mu_0}(\Gamma)} F\mathcal{L}^* f = 0, \quad \mu_0 = 1, m_1, \tag{4.11}$$

$\forall$ vector fields Z_μ s.t. $S(Z_\mu) = K^v_\mu$ and $\forall Y_{\mu_0}$ s.t. $S(Y_{\mu_0}) = R(L)X_{\phi^{(1)}_{\mu_0}}$.

If the Lagrangian is not degenerate it is well known that a dynamical symmetry corresponds to each constant of the motion. In fact, when $\omega_{\mathcal{L}}$ is a symplectic form, it is always possible, given $G_{\mathcal{L}} \in \mathcal{F}(TQ)$, to solve the equation for $X \in \mathcal{X}(TQ)$,

$$i_X \omega_{\mathcal{L}} = dG_{\mathcal{L}}. \tag{4.12}$$

Then, if $G_{\mathcal{L}}$ is a constant of the motion one gets

$$i_{[\Gamma,X]}\omega_{\mathcal{L}} = L_\Gamma i_X \omega_{\mathcal{L}} = dL_\Gamma G_{\mathcal{L}} = 0, \tag{4.13}$$

that is to say

$$[X,\Gamma] = 0, \tag{4.14}$$

which is taken as a definition of dynamical symmetry. As a consequence of (4.14), every dynamical symmetry is a Newtonoid vector field.

In order to generalize this result to degenerate systems, let us recall that in such a case a constant of the motion satisfies

$$j^*_{S_2} L_\Gamma G_{\mathcal{L}} = 0, \tag{4.15}$$

where Γ is the second order dynamics tangent to S_2. As Γ is determined except for arbitrary elements of $\underline{\mathcal{X}(S_2)}^\perp \cap V(TQ)$, from (4.15) it can be inferred that a constant of the motion $G \in \mathcal{F}(T^*Q)$ exists such that

$$j^*_{S_2}(G_{\mathcal{L}} - F\mathcal{L}^* G) = 0. \tag{4.16}$$

Since one can always choose G (see ref. [2]) so as to be first class on M_1, it is always possible (see ref. [16]) to solve the equation for $X \in \mathcal{X}(TQ)$,

$$j^*_{S_2} i_X \omega_{\mathcal{L}} = j^*_{S_2} dG_{\mathcal{L}}, \tag{4.17}$$

from which it immediately follows that, if $X \in \underline{\mathcal{X}(S_2)}$,

$$j^*_{S_2} i_{[X,\Gamma]}\omega_{\mathcal{L}} = 0. \tag{4.18}$$

However, in general the field $[X,\Gamma] \in \underline{\mathcal{X}(S_2)}$ is not a vertical field. So we need to impose that a Newtonoid solution of (4.17) exists.

Since it is always possible, as shown in ref. [16], to find such a solution among the fields of $\underline{\mathcal{X}(S_2)}$, one can always associate a dynamical symmetry with each Lagrangian constant of the motion. As we have seen in section 2, the same result holds in T^*Q with the corresponding first class constants of the motion, only if we accept Dirac's conjecture. Otherwise, one gets conditions (2.37), (2.38) and a constant of the motion does not necessarily generate a DST. So, if one follows the formulation given in ref. [7] the one-to-one correspondence between Hamiltonian and Lagrangian DST seems to be broken because each first class constant of the motion G is such that $F\mathcal{L}^*G$, regarded as a constant of the motion in TQ, is associated with a Lagrangian symmetry.

However, one can find a one-to-one connection between DST defined as in ref. [7] and a particular kind of dynamical symmetry vector fields of $\mathcal{X}(TQ)$. In what follows it will be demonstrated that imposing the $F\mathcal{L}$-projectability of the first n components of a Lagrangian DST, one achieves transformations whose push-forward in the phase space is an infinitesimal canonical transformation with a generator G satisfying (2.37), (2.38).

In order to do this, an $F\mathcal{L}$-projectable solution of (4.17) is the field $\overline{X}_{(G)} \in \underline{\mathcal{X}(S_2)}$ defined in section 3,

$$j^*_{S_2} i_{\overline{X}_{(G)}} \omega_{\mathcal{L}} = j^*_{S_2} dG_{\mathcal{L}}. \tag{4.19}$$

For what has been mentioned above, if the Newtonoid vector field

$$X_{(G)}(\Gamma) = \overline{X}_{(G)} + S[\Gamma, \overline{X}_{(G)}] \tag{4.20}$$

is a solution of (4.17) and is tangent to S_2, then it is a Lagrangian DST. Because of (4.19), the following proposition evidently holds true:

Proposition 4.3. *A necessary and sufficient condition for*

$$j^*_{S_2} i_{X_{(G)}(\Gamma)} \omega_{\mathcal{L}} = j^*_{S_2} dG_{\mathcal{L}}, \tag{4.21}$$

$$X_{(G)}(\Gamma) \in \underline{\mathcal{X}(S_2)}, \tag{4.22}$$

is that

$$S[\Gamma, \overline{X}_{(G)}] \in \underline{\mathcal{X}(S_2)}^{\perp}. \tag{4.23}$$

One can derive an equivalent condition which clarifies the link with dynamical symmetries of T^*Q. Using the property (see ref. [15])

$$R(L) F\mathcal{L}_* X = S(X), \tag{4.24}$$

where $X \in \mathcal{X}(TQ)$ is an $F\mathcal{L}$-projectable vector field, along with the property (3.39), one has

$$S[\Gamma, \overline{X}_{(G)}]\big|_{S_2} = R(L) X_{\{G, H^{(2)}\}}\big|_{S_2} + (j^*_{S_2} v^{\nu_1}) R(L) X_{\{G, \phi^{(0)}_{\nu_1}\}}\big|_{S_2}. \tag{4.25}$$

From this expression it is easy to see that (2.37), (2.38) imply (4.23) and so, for every DST in T^*Q, a Lagrangian DST exists.

Explicitly ruling out the possibility that the dimensions of Q are reduced by the equations of motion, we can demonstrate the following

Proposition 4.4. *The vector field $S[\Gamma, \overline{X}_{(G)}]$ belongs to $\underline{\mathcal{X}(S_2)}^{\perp}$ if and only if*

$$R(L)X_{\{G,H^{(2)}\}} \in \underline{\mathcal{X}(S_2)}^{\perp}, \tag{4.26}$$

$$R(L)X_{\{G,\phi^{(0)}_{\mu_1}\}} \in \underline{\mathcal{X}(S_2)}^{\perp}, \quad \mu_1 = 1, m_2. \tag{4.27}$$

Proof. The sufficiency of conditions (4.26) and (4.27) is evident for (4.25). Let us suppose, conversely, that (4.23) holds true. We have

$$j^*_{S_2} i_{S[\Gamma,\overline{X}_{(G)}]}\omega_{\mathcal{L}} = 0. \tag{4.28}$$

Taking into account the property [14]

$$i_{S(X)}\omega_{\mathcal{L}} = -(i_X\omega_{\mathcal{L}}) \circ S, \tag{4.29}$$

we can write

$$i_{R(L)X_f}\omega_{\mathcal{L}} = -(dF\mathcal{L}^* f) \circ S, \quad \forall f \in \mathcal{F}(T^*Q). \tag{4.30}$$

Therefore, from (4.25) and (4.28) one gets

$$j^*_{S_2}((dF\mathcal{L}^*\{G, H^{(2)}\}) \circ S + v^{\mu_1}(dF\mathcal{L}^*\{G, \phi^{(0)}_{\mu_1}\}) \circ S) = 0. \tag{4.31}$$

This means that $\forall V \in V(TQ)$,

$$j^*_{S_2}(L_V(F\mathcal{L}^*\{G, H^{(2)}\}) + v^{\mu_1}L_V(F\mathcal{L}^*\{G, \phi^{(0)}_{\mu_1}\})) = 0. \tag{4.32}$$

Differentiating (4.32) with respect to the field $K^v_{\mu_1}$, which is tangent to S_2, and using (3.22) one obtains for $\mu_1 = 1, m_2$,

$$j^*_{S_2}(L_{[K^v_{\mu_1},V]}(F\mathcal{L}^*\{G, H^{(2)}\}) \quad + v^{\nu_1}(L_{[K^v_{\mu_1},V]}(F\mathcal{L}^*\{G, \phi^{(0)}_{\nu_1}\})$$
$$+ L_V(F\mathcal{L}^*\{G, \phi^{(0)}_{\mu_1}\}))) = 0. \tag{4.33}$$

Since (4.32) holds true and $[K^v_{\mu_1}, V]$ is vertical, it can be inferred that

$$j^*_{S_2} L_V(F\mathcal{L}^*\{G, \phi^{(0)}_{\mu_1}\}) = 0, \quad \forall V \in V(TQ); \quad \mu_1 = 1, m_2, \tag{4.34}$$

and, as a consequence,

$$j^*_{S_2} L_V(F\mathcal{L}^*\{G, H^{(2)}\}) = 0, \quad \forall V \in V(TQ). \tag{4.35}$$

Using (4.30) once more one immediately sees that

$$R(L)X_{\{G,H^{(2)}\}} \in \mathcal{X}(S_2)^{\perp}, \tag{4.36}$$

$$R(L)X_{\{G,\phi^{(0)}_{\mu_1}\}} \in \mathcal{X}(S_2)^{\perp}, \quad \mu_1 = 1, m_2. \tag{4.37}$$

It remains to be proved that these vector fields are also tangent to S_2. Since by hypothesis

$$j^*_{S_2} F\mathcal{L}^*\{G, H^{(2)}\} = 0, \tag{4.38}$$

$$j^*_{S_2} F\mathcal{L}^*\{G, \phi^{(0)}_{\mu_1}\} = 0, \quad \mu_1 = 1, m_2, \tag{4.39}$$

properties (4.34) and (4.35) imply that $\forall X \in \mathcal{X}(TQ)$,

$$j^*_{S_2} L_X (F\mathcal{L}^*\{G, H^{(2)}\}) = 0, \tag{4.40}$$

$$j^*_{S_2} L_X (F\mathcal{L}^*\{G, \phi^{(0)}_{\mu_1}\}) = 0, \quad \forall V \in V(TQ); \quad \mu_1 = 1, m_2. \tag{4.41}$$

Keeping this result in mind, lemma 4.2 can be used to prove that the vector fields of conditions (4.26), (4.27) are tangent to S_2. □

Now conditions (4.26) and (4.27) are completely equivalent to conditions (2.37) and (2.38). Finally, one can take a look into the particular case examined at the end of section 2: when the dynamical constraint $F\mathcal{L}^*\overline{\phi}^{(1)}_{\mu_1}$ is an identity in TQ, the Lagrangian DST corresponding to $X_{\phi^{(0)}_{\mu_1}}$ is the field $K_{\mu_1}(\Gamma)$. Moreover, one immediately obtains a symmetry for the Lagrangian. In fact, from the expression for primary Lagrangian constraints shown in ref. [17], it follows that

$$L_{K_{\mu_1}(\Gamma_0)}\mathcal{L} = L_{\Gamma_0}(L_{K^v_{\mu_1}}\mathcal{L}), \quad \forall \Gamma_0 : S(\Gamma_0) = \Delta. \tag{4.42}$$

Such symmetries are always present in degenerate systems free of constraints, as the free relativistic particle and the electron–monopole system.

5. Examples

Example 1. Let us consider the Lagrangian function

$$\mathcal{L} = \tfrac{1}{2}(q_1\dot{q}_1 + q_2\dot{q}_2)^2 + 2q_2\dot{q}_3(q_1\dot{q}_1 + q_2\dot{q}_2) + 2q_2^2\dot{q}_3^2 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 + q_1\dot{q}_4.$$

If we think that the chart in which we are working is such that q_1 is different from 0, then the Hessian matrix (2.1) has constant rank equal to 1. Such a prescription, as will be seen, is consistent with the solutions of the equations of motion. Indeed, we limit the study to the sector in which q_2 is different from zero.

The Hamiltonian analysis of the constraints starts from the function

$$H = p\dot{q} - \mathcal{L} = \frac{1}{2}\left(\frac{p_1}{q_1} - 1\right)^2.$$

It is easy to verify that the primary constraints

$$\phi_1^{(0)} = 2(p_2 - q_2) - (p_3 - q_3), \quad \phi_2^{(0)} = q_2 p_1 - p_2 q_1, \quad \phi_3^{(0)} = p_4 - q_1$$

(only $\phi_1^{(0)}$ being first class) are independent. Moreover the multipliers $\lambda_2 = (q_1 - p_1)/q_1^2 q_2$, $\lambda_3 = 0$ are determined on M_0 so that, recalling (2.8), we can write

$$H^{(1)} = \frac{p_1 - q_1}{q_1}\left(\frac{p_2}{q_2} - \frac{p_1 + q_1}{2q_1}\right),$$

$$\overline{\phi}_1^{(1)} = \{\phi_1^{(0)}, H^{(1)}\} = \frac{2}{q_2}\frac{p_1 - q_1}{q_1}\frac{p_2 - q_2}{q_2}.$$

As this constraint is quadratic, the conditions (2.37) and (2.38) on $\phi_1^{(0)}$ are satisfied. On the other hand, if one looks for a non-quadratic first class constraint, we note that it is enough to take $\phi_1^{(1)} = p_2 - q_2$. By imposing the stability condition to $\phi_1^{(1)}$ there are no further constraints.

On TQ, on the other hand, a basis for $\ker \omega_{\mathcal{L}}$ is

$$K_1 = \frac{\partial}{\partial \dot{q}_2} - \frac{q_2}{q_1}\frac{\partial}{\partial \dot{q}_1}, \qquad K_2 = \frac{\partial}{\partial \dot{q}_3} - \frac{2q_2}{q_1}\frac{\partial}{\partial \dot{q}_1}, \qquad K_3 = \frac{\partial}{\partial \dot{q}_4},$$

$$K_4 = 2\frac{\partial}{\partial q_2} - \frac{\partial}{\partial q_3} - 2\frac{q_1}{q_2}[q_1\dot{q}_1 + q_2(\dot{q}_2 + 2\dot{q}_3)]\frac{\partial}{\partial q_4} - \frac{2}{q_1 q_2}[q_1\dot{q}_1 + 2q_2(\dot{q}_2 + 2\dot{q}_3)]\frac{\partial}{\partial \dot{q}_1}.$$

There are two SODE constraints $\chi_1' = \dot{q}_1$, $\chi_2' = \dot{q}_4$, and only one dynamical constraint $\chi = -L_{K_4} E_{\mathcal{L}}$,

$$\chi = \frac{2}{q_2}[q_1\dot{q}_1 + q_2(\dot{q}_2 + 2\dot{q}_3)]^2 = F\mathcal{L}^*\overline{\phi}_1^{(1)}.$$

A second order solution of the dynamical equation on S_1 is

$$\Gamma = \dot{q}_1\frac{\partial}{\partial q_1} + \dot{q}_2\frac{\partial}{\partial q_2} + \dot{q}_3\frac{\partial}{\partial q_3} + \dot{q}_4\frac{\partial}{\partial q_4} + \eta_1 K_1 + \eta_2 K_2 + \eta_3 K_3,$$

with arbitrary $\eta_1, \eta_2, \eta_3 \in \mathcal{F}(TQ)$.

By linearizing χ and imposing the tangency of Γ to S_1, we get $\eta_1 + 2\eta_2 = 0$, $\eta_3 = 0$, whereas there are no secondary Lagrangian constraints. So, the final dynamics $\Gamma \in \underline{\mathcal{X}(S_1)}$ is

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q_i} + \eta \left(\frac{\partial}{\partial \dot{q}_3} - 2\frac{\partial}{\partial \dot{q}_2} \right),$$

with an arbitrary function η multiplying a vector field belonging to $V(\ker \omega_{\mathcal{L}}) \cap \underline{\mathcal{X}(S_1)}$. We note that q_1 is constant; therefore, if it is different from zero for $t = 0$, then it never vanishes. Finally, we find that $K_4 \in \underline{\mathcal{X}(S_1)}$ and the vector field $K_4(\Gamma)$ is such that

$$K_4(\Gamma)\big|_{S_1} = K_4\big|_{S_1} = \left(2\frac{\partial}{\partial q_2} - \frac{\partial}{\partial q_3} \right)\Big|_{S_1}.$$

Indeed we obtain

$$[K_4(\Gamma), \Gamma]\big|_{S_1} = (j^*_{S_1} L_{K_4(\Gamma)} \eta) \left(2\frac{\partial}{\partial \dot{q}_2} - \frac{\partial}{\partial \dot{q}_3} \right)\Big|_{S_1} \in V(\ker \omega_{\mathcal{L}}) \cap \underline{\mathcal{X}(S_1)}.$$

The field $K_4(\Gamma)$ is a Lagrangian DST and its first four components are $F\mathcal{L}$-projectable; consequently, as we have just seen, the constraint $\phi_1^{(0)}$ is the generator of a Hamiltonian DST, defined as in ref. [4].

Example 2. As in ref. [18] we consider

$$\mathcal{L} = \tfrac{1}{2}[\dot{q}_1^2 + q_1^2(\dot{q}_2 - q_3)^2] - V(q_1).$$

In this case a basis for $\ker \omega_{\mathcal{L}}$ is given by

$$K_1 = \partial/\partial q_3 + \partial/\partial \dot{q}_2; \quad K_2 = \partial/\partial \dot{q}_3.$$

There is only a dynamical constraint $\chi = q_1^2(\dot{q}_2 - q_3)$, to which the second order dynamics

$$\Gamma = \dot{q}_i \frac{\partial}{\partial q_i} + \left[q_1(\dot{q}_2 - q_3)^2 - \frac{\mathrm{d}V}{\mathrm{d}q_1} \right] \frac{\partial}{\partial \dot{q}_1} + \left[\dot{q}_3 - \frac{2}{q_1}(\dot{q}_2 - q_3)\dot{q}_1 \right] \frac{\partial}{\partial \dot{q}_2} + \eta \frac{\partial}{\partial \dot{q}_3}$$

is automatically tangent; η represents the residual arbitrariness. We get a basis for the orthogonal complement $\underline{\mathcal{X}(S_1)}^{\perp}$ by adding to K_1 and K_2 the vector field $Y = \partial/\partial q_2$. The condition of proposition 4.4 holds: $S[\Gamma, Y] = 0$. It is easy to see that

$$Y(\Gamma) \in \underline{\mathcal{X}(S_1)}, \qquad [Y(\Gamma), \Gamma] \in V(\ker \omega_{\mathcal{L}}) \cap \underline{\mathcal{X}(S_1)}.$$

We would like to underline that for this example the condition given in ref. [4] holds: $K(\overline{\phi}^{(1)}_{\mu_1})$ is strongly equal to zero on S_1, where $\overline{\phi}^{(1)}_{\mu_1}$ is in our case equal to p_2. This condition ensures the existence of a Hamiltonian DST in the sense specified in ref. [4].

Example 3. As found in ref. [17], the Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{q}_1^2 + q_1^2\dot{q}_2^2 + q_1^2\dot{q}_3^2 \sin^2 q_2) + n\left(2\frac{\dot{q}_4}{g} + (\cos q_2 - 1)\dot{q}_3 + \frac{q_4}{g}\frac{\sin q_2}{1 + \cos q_2}\dot{q}_2\right),$$

with $g(q_2, q_4) = [\frac{1}{2}(\cos q_2 + 1) - q_4^2]^{1/2}$ describes the motion of a nonrelativistic charged particle in the field of a magnetic monopole. As in the case of the relativistic free particle, here Lagrangian constraints do not exist and the Newtonoid symmetry is

$$K(\Gamma) = \eta \frac{\partial}{\partial q_4} + (L_\Gamma \eta)\frac{\partial}{\partial \dot{q}_4},$$

with η being arbitrary.

6. Conclusions

In the present work an intrinsic definition has been given for DST in TQ. To do so, specific vector fields of $\mathcal{X}(TQ)$ have been identified, which play a role analogous to that of Hamiltonian fields; indeed, particular care has been devoted to the problem of their tangency to the final constraint submanifold.

A proof has been given of the fact that a DST on TQ corresponds to every DST in T^*Q. The converse is not true if one refuses Dirac's conjecture, because in such a case there are first class constants of motion in T^*Q which do not generate any DST.

Appendix A. Local expressions for K_{μ_0} and $X_{(G)}$

The local expression of $\omega_\mathcal{L}$ is

$$\omega_\mathcal{L} = W_{ij}\,\mathrm{d}q^i \wedge \mathrm{d}\dot{q}^j + \tfrac{1}{2}A_{ij}\,\mathrm{d}q^i \wedge \mathrm{d}q^j \tag{A.1}$$

with

$$W_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}, \qquad A_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial q^j} - \frac{\partial^2 \mathcal{L}}{\partial q^i \partial \dot{q}^j}, \quad i,j = 1,n.$$

While the vector fields which satisfy (3.5) must evidently have the following form

$$K^v_\mu = \left(F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_i}\right) \frac{\partial}{\partial \dot{q}^i}, \quad \mu = 1,m, \tag{A.2}$$

the fields $K_{\mu_0} \in \ker \omega_\mathcal{L}$, which we write as

$$K_{\mu_0} = a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i}, \quad \mu_0 = 1,m_1,$$

must satisfy the condition

$$W_{ij} b^j + A_{ij} a^j = 0, \quad i = 1,n.$$

Consequently, if one tries to obtain

$$F\mathcal{L}_* K_{\mu_0} = X_{\phi^{(0)}_{\mu_0}}, \quad \mu_0 = 1,m_1,$$

the n equations for b^j remain to be solved:

$$W_{ij} b^j = A_{ji} F\mathcal{L}^* \frac{\partial \phi^{(0)}_{\mu_0}}{\partial p_j}. \tag{A.3}$$

In order to express the b^j components use will be made of the important completeness relationship (see ref. [16]):

$$W_{ij}\left(F\mathcal{L}^* \frac{\partial^2 H}{\partial p_i \partial p_k} + v^\mu F\mathcal{L}^* \frac{\partial^2 \phi^{(0)}_\mu}{\partial p_i \partial p_k}\right)$$
$$= \delta^j_k - \frac{\partial v^\mu}{\partial \dot{q}^j} F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_k}, \quad j,k = 1,n, \tag{A.4}$$

from which, considering the Lagrangian identity

$$\dot{q}^i = F\mathcal{L}^* \frac{\partial H}{\partial p_i} + v^\mu F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_i}, \quad i = 1,n, \tag{A.5}$$

and the fact that

$$F\mathcal{L}^*\phi_\mu^{(0)} = 0, \quad \mu = 1, m,$$

we get

$$\begin{aligned} b^k - b^j \frac{\partial v^\mu}{\partial \dot{q}^j} F\mathcal{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p_k} &= F\mathcal{L}^* \frac{\partial \phi_{\mu_0}^{(0)}}{\partial p_j} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^j \partial q^i} - \frac{\partial^2 \mathcal{L}}{\partial q^j \partial \dot{q}^i} \right) \left(F\mathcal{L}^* \frac{\partial^2 H}{\partial p_i \partial p_k} + v^\mu F\mathcal{L}^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial p_i \partial p_k} \right) \\ &= -F\mathcal{L}^* \frac{\partial \phi_{\mu_0}^{(0)}}{\partial q^j} \left(F\mathcal{L}^* \frac{\partial^2 H}{\partial p_j \partial p_k} + v^\mu F\mathcal{L}^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial p_j \partial p_k} \right) \\ &\quad + F\mathcal{L}^* \frac{\partial \phi_{\mu_0}^{(0)}}{\partial p_j} \left(F\mathcal{L}^* \frac{\partial^2 H}{\partial q^j \partial p_k} + v^\mu F\mathcal{L}^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial q^j \partial p_k} \right), \end{aligned}$$

for which we obtain

$$\begin{aligned} b^k &= F\mathcal{L}^* \left\{ \frac{\partial H}{\partial p_k}, \phi_{\mu_0}^{(0)} \right\} + v^\mu F\mathcal{L}^* \left\{ \frac{\partial \phi_\mu^{(0)}}{\partial p_k}, \phi_{\mu_0}^{(0)} \right\} \\ &\quad + b^j \frac{\partial v^\mu}{\partial \dot{q}^j} F\mathcal{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p_k}, \quad k = 1, n. \end{aligned}$$

Since the matrix W is singular we can add the corresponding components of any null eigenvector of W to the b^k; such an arbitrariness is taken into account by finally writing

$$\begin{aligned} K_{\mu_0} &= F\mathcal{L}^* \frac{\partial \phi_{\mu_0}^{(0)}}{\partial p_i} \frac{\partial}{\partial q^i} \\ &\quad + \left[F\mathcal{L}^* \left\{ \frac{\partial H}{\partial p_i}, \phi_{\mu_0}^{(0)} \right\} + v^\mu F\mathcal{L}^* \left\{ \frac{\partial \phi_\mu^{(0)}}{\partial p_i}, \phi_{\mu_0}^{(0)} \right\} + a^\mu F\mathcal{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p_i} \right] \frac{\partial}{\partial \dot{q}^i}, \end{aligned} \tag{A.6}$$

with a^μ arbitrary functions. Differentiating (A.5) with respect to K_{μ_0} and comparing the result with b^i, we obtain

$$L_{K_{\mu_0}} v^\mu = a^\mu, \quad \mu = 1, m. \tag{A.7}$$

Therefore (3.15) is equivalent to the choice $a^\mu = 0$ to remove all arbitrariness. Indeed, let $G \in \mathcal{F}(T^*Q)$ be such that $j^*_{M_1}\{G, \phi_\mu^{(0)}\} = 0$, with $\mu = 1, m$. In the same way adopted to derive (A.6) one can easily find that

$$X_{(G)} = F\mathcal{L}^* \frac{\partial G}{\partial p_i}\frac{\partial}{\partial q^i} + \left[F\mathcal{L}^* \left\{ \frac{\partial H}{\partial p_i}, G \right\} + v^\mu F\mathcal{L}^* \left\{ \frac{\partial \phi^{(0)}_\mu}{\partial p_i}, G \right\} + F\mathcal{L}^* \{G, \varphi_\rho\} M_{\rho i} \right] \frac{\partial}{\partial \dot{q}^i}, \tag{A.8}$$

where the φ_ρ's ($\rho = n - m + 1, n$) make up a set of primary constraints which are equivalent to the $\phi^{(0)}_\mu$'s ($\mu = 1, m$), and satisfy (see ref. [19])

$$\phi^{(0)}_\mu \underset{M_0}{\equiv} \frac{\partial \phi^{(0)}_\mu}{\partial p_\rho} \varphi_\rho, \quad \mu = 1, m, \tag{A.9}$$

and where

$$M_{ik} = F\mathcal{L}^* \frac{\partial^2 H}{\partial p_i \partial p_k} + v^\mu F\mathcal{L}^* \frac{\partial^2 \phi^{(0)}_\mu}{\partial p_i \partial p_k}, \quad i, k = 1, n.$$

Appendix B. How to obtain a Lie bracket $[\Gamma, X_{(G)}]$

Here we wish to calculate the local expression of

$$\left[\Gamma, \overline{K}_{\mu_1}\right] = a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i}. \tag{B.1}$$

Here no detailed form of Γ will be used but rather the action of the operator K, eq. (3.4), and property (3.27) will be employed.

Thus in a few steps we obtain the first components of the field resticted to S_2:

$$a^i \underset{S_2}{=} F\mathcal{L}^* \frac{\partial}{\partial p_i}\{\phi^{(0)}_{\mu_1}, H\} v^\mu F\mathcal{L}^* \frac{\partial}{\partial p_i}\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_\mu\} - F\mathcal{L}^* \left[\{\lambda^{\nu'_0}, \phi^{(0)}_{\mu_1}\} \frac{\partial \phi^{(0)}_{\nu'_0}}{\partial p_i} + \{\lambda^{\nu'_1}, \phi^{(0)}_{\mu_1}\} \frac{\partial \phi^{(0)}_{\nu'_1}}{\partial p_i} \right], \quad i = 1, n.$$

Let us consider the previously quoted constraint expression

$$j^*_{S_2}(v^{\nu'_0} - F\mathcal{L}^* \lambda^{\nu'_0}) = 0, \quad \nu'_0 = 1, m - m_1, \tag{B.2}$$

$$j^*_{S_2}(v^{\nu'_1} - F\mathcal{L}^* \lambda^{\nu'_1}) = 0, \quad \nu'_1 = 1, m - m_2. \tag{B.3}$$

By splitting the v^{μ}'s and considering $H^{(2)}$ as defined in the first section, we have

$$a^i \underset{S_2}{=} F\mathcal{L}^* \frac{\partial}{\partial p_i}\{\phi^{(0)}_{\mu_1}, H^{(2)}\} + v^{\nu_1} F\mathcal{L}^* \frac{\partial}{\partial p_i}\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}, \quad i = 1, n. \tag{B.4}$$

Again, with the aid of (3.27) and, moreover, taking into account the Jacobi identity, we can write

$$\begin{aligned} b^i \underset{S_2}{=} & F\mathcal{L}^* \left\{\frac{\partial H}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, H\}\right\} + v^{\mu} F\mathcal{L}^* \left\{\frac{\partial H}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\mu}\}\right\} \\ & + v^{\mu} F\mathcal{L}^* \left\{\frac{\partial \phi^{(0)}_{\mu}}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, H\}\right\} + v^{\mu} v^{\nu} F\mathcal{L}^* \left\{\frac{\partial \phi^{(0)}_{\mu}}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu}\}\right\} \\ & + K \left(\{\lambda^{\nu'_0}, \phi^{(0)}_{\mu_1}\} \frac{\partial \phi^{(0)}_{\nu'_0}}{\partial p_i} + \{\lambda^{\nu'_1}, \phi^{(0)}_{\mu_1}\} \frac{\partial \phi^{(0)}_{\nu'_1}}{\partial p_i}\right) \\ & - L_{\overline{K}_{\mu_1}} v^{\mu} F\mathcal{L}^* \left\{\frac{\partial H}{\partial p_i}, \phi^{(0)}_{\mu}\right\} - (L_{\overline{K}_{\mu_1}} L_r v^{\mu}) F\mathcal{L}^* \frac{\partial \phi^{(0)}_{\mu}}{\partial p_i} \\ & - (L_{\overline{K}_{\mu_1}} v^{\mu}) L_r F\mathcal{L}^* \frac{\partial \phi^{(0)}_{\mu}}{\partial p_i} - v^{\nu} (L_{\overline{K}_{\mu_1}} v^{\mu}) F\mathcal{L}^* \left\{\frac{\partial \phi^{(0)}_{\nu}}{\partial p_i}, \phi^{(0)}_{\mu}\right\}, \quad i = 1, n. \end{aligned}$$

Again, splitting the functions v^{μ}, taking into account (B.2) and recalling proposition 3.1, with the same easy but boring calculations we finally obtain, for $i = 1, n$:

$$\begin{aligned} b^i \underset{S_2}{=} & F\mathcal{L}^* \left\{\frac{\partial H}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, H^{(2)}\}\right\} + v^{\mu} F\mathcal{L}^* \left\{\frac{\partial \phi^{(0)}_{\mu}}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, H^{(2)}\}\right\} \\ & + F\mathcal{L}^* \{\lambda^{\nu'_0}, \{\phi^{(0)}_{\mu_1}, H^{(2)}\}\} \frac{\partial \phi^{(0)}_{\nu'_0}}{\partial p_i} + F\mathcal{L}^* \{\lambda^{\nu'_1}, \{\phi^{(0)}_{\mu_1}, H^{(2)}\}\} \frac{\partial \phi^{(0)}_{\nu'_1}}{\partial p_i} \\ & + v^{\nu_1} \left(F\mathcal{L}^* \left\{\frac{\partial H}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}\right\} + v^{\mu} F\mathcal{L}^* \left\{\frac{\partial \phi^{(0)}_{\mu}}{\partial p_i}, \{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}\right\}\right) \\ & + v^{\nu_1} F\mathcal{L}^* \left(\{\lambda^{\nu'_0}, \{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}\} \frac{\partial \phi^{(0)}_{\nu'_0}}{\partial p_i} + \{\lambda^{\nu'_1}, \{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}\} \frac{\partial \phi^{(0)}_{\nu'_1}}{\partial p_i}\right) \\ & - \left(L_{\overline{K}_{\mu_1}} L_r v^{\nu_1}\right) F\mathcal{L}^* \frac{\partial \phi^{(0)}_{\nu_1}}{\partial p_i}. \end{aligned} \tag{B.5}$$

Now, analyzing (B.4) and (B.5) some previously defined vector fields can be recognized. In fact, the field whose first n components are

$$F\mathcal{L}^* \frac{\partial}{\partial p_i}\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\},$$

belongs to $\ker\omega_{\mathcal{L}} \cap \underline{\mathcal{X}(S_2)}$. This can be seen by comparing (3.36) and (A.6) with (B.5), since the function $\{\phi_{\mu_1}^{(0)}, \phi_{\nu_1}^{(0)}\}$ is again a primary first class constraint on M_1 . Moreover, a linear combination of fields belonging to $V(\ker\omega_{\mathcal{L}}) \cap \underline{\underline{\mathcal{X}(S_2)}}$ is present. Thus we are left with a vector field which is indicated by $\overline{Y}_{\mu_1}$. Its local expression is consequently

$$\begin{aligned}
\overline{Y}_{\mu_1} &= F\mathcal{L}^* \frac{\partial \overline{\phi}^{(1)}_{\mu_1}}{\partial p_i} \frac{\partial}{\partial q^i} \\
&+ \left[F\mathcal{L}^* \left\{ \frac{\partial H}{\partial p_i}, \overline{\phi}^{(1)}_{\mu_1} \right\} + v^\mu F\mathcal{L}^* \left\{ \frac{\partial \phi^{(0)}_\mu}{\partial p_i}, \overline{\phi}^{(1)}_{\mu_1} \right\} \right] \frac{\partial}{\partial \dot{q}^i} \\
&+ F\mathcal{L}^* \left[\{\lambda^{\mu'_0}, \overline{\phi}^{(1)}_{\mu_1}\} \frac{\partial \phi^{(0)}_{\mu'_0}}{\partial p_i} + \{\lambda^{\mu'_1}, \overline{\phi}^{(1)}_{\mu_1}\} \frac{\partial \phi^{(1)}_{\mu'_1}}{\partial p_i} \right] \frac{\partial}{\partial \dot{q}^i},
\end{aligned} \tag{B.6}$$

where $\overline{\phi}^{(1)}_{\mu_1} = \{\phi^{(0)}_{\mu_1}, H^{(2)}\}$.

Appendix C. Proof of lemma 4.2

Taking the functions $K(\phi^{(0)}_\mu)$ and $K(\phi^{(1)}_{\mu_0})$ as Lagrangian constraints, let us calculate

$$\begin{aligned}
&L_{R(L)X_f} K(\phi^{(0)}_\mu) \\
&= L_{R(L)X_f} \left(\dot{q}^i F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial q^i} + \frac{\partial \mathcal{L}}{\partial q^i} F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_i} \right) \\
&= F\mathcal{L}^* \left(\frac{\partial f}{\partial p_i} \frac{\partial \phi^{(0)}_\mu}{\partial q^i} \right) + \dot{q}^i \left(F\mathcal{L}^* \frac{\partial f}{\partial p_k} \right) \frac{\partial}{\partial \dot{q}^k} \left(F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial q^i} \right) \\
&\quad + \left(F\mathcal{L}^* \frac{\partial f}{\partial p_k} \right) \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial q^i} \left(F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_i} \right) \\
&\quad + \frac{\partial \mathcal{L}}{\partial q^i} \left(F\mathcal{L}^* \frac{\partial f}{\partial p_k} \right) \frac{\partial}{\partial \dot{q}^k} \left(F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_i} \right) \\
&= F\mathcal{L}^* \left(\frac{\partial f}{\partial p_i} \frac{\partial \phi^{(0)}_\mu}{\partial q^i} \right) + \left(\frac{\partial}{\partial q^i} F\mathcal{L}^* f \right) F\mathcal{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p_i} - F\mathcal{L}^* \left(\frac{\partial f}{\partial q^i} \frac{\partial \phi^{(0)}_\mu}{\partial p_i} \right) \\
&\quad + \dot{q}^i \left(F\mathcal{L}^* \frac{\partial f}{\partial p_k} \right) \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial \dot{q}^j} \left(F\mathcal{L}^* \frac{\partial^2 \phi^{(0)}_\mu}{\partial p_j \partial q^i} \right) \\
&\quad + \frac{\partial \mathcal{L}}{\partial q^i} \left(F\mathcal{L}^* \frac{\partial f}{\partial p_k} \right) \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial \dot{q}^j} \left(F\mathcal{L}^* \frac{\partial^2 \phi^{(0)}_\mu}{\partial p_j \partial p_i} \right)
\end{aligned}$$

$$= F\mathcal{L}^*\{\phi_\mu^{(0)}, f\} + F\mathcal{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p_i} \frac{\partial}{\partial q^i} F\mathcal{L}^* f + K \left(\frac{\partial \phi_\mu^{(0)}}{\partial p_i} \right) \frac{\partial}{\partial \dot{q}^i} (F\mathcal{L}^* f)$$

$$\underset{S_2}{=} F\mathcal{L}^*\{\phi_\mu^{(0)}, f\} + L_{Z_\mu(\Gamma)} (F\mathcal{L}^* f).$$

Property (3.27) was used in the last passage. In a similar manner an analogous result can be obtained,

$$L_{R(L)X_f} K(\phi_{\mu_0}^{(1)}) \underset{S_2}{=} F\mathcal{L}^*\{\phi_{\mu_0}^{(1)}, f\} + L_{Y_{\mu_0}(\Gamma)} (F\mathcal{L}^* f).$$

If f is a first class function, we have

$$j_{S_1'}^* F\mathcal{L}^*\{\phi_\mu^{(0)}, f\} = F\mathcal{L}_{S_1'}^* j_{M_1}^* \{\phi_\mu^{(0)}, f\}, \quad \mu = 1, m,$$

$$j_{S_1'}^* F\mathcal{L}^*\{\phi_{\mu_0}^{(1)}, f\} = F\mathcal{L}_{S_1'}^* j_{M_1}^* \{\phi_{\mu_0}^{(1)}, f\}, \quad \mu_0 = 1, m_1.$$

By requiring the tangency of the vertical field, lemma 4.2 follows.

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